Robustness Optimization of Structural and Controller Parameters

Kyong B. Lim*
Virginia Polytechnic Institute and State University, Blacksburg, Virginia
and
John L. Junkins†
Texas A&M University, College Station, Texas

In this paper, several novel ideas in the definition and optimization of robustness for structures and structural controllers are presented. A robustness bound attributable to Patel and Toda is developed using eigenvalue conditioning analysis. Homotopy and sequential linear programming algorithms are used in lieu of conventional nonlinear programming to implement these ideas for an illustrative example. The numerical results confirm the conservatism of the stability robustness bound but nevertheless support the hypothesis that maximizing the robustness measure does significantly increase the true robustness of a closed-loop system. The results also indicate that maximizing the stability robustness measure produces more robust designs than minimizing eigenvalue sensitivity directly.

I. Introduction

FOR some proposed space structures[1,2] it is not difficult to imagine that some prescribed performance requirements may give rise to a scenario where an ad hoc structure design is not optimally suited for a controller design, or where a feasible controller that satisfies a set of a priori desired constraints may not even exist, especially for marginally feasible missions. One may then wish to investigate a unified approach whereby structure and controllers are designed simultaneously to achieve improved performance.

Some recent work on the unified approach includes Refs. 3–10. To the authors' knowledge, the first general simultaneous optimization formulation is given in Ref. 7. Here, a minimum modification strategy in conjunction with a continuation method[11] of handling constraint equations is used for eigenvalue optimizations. Although the results in Refs. 7 and 8 demonstrated an algorithm with convergence to moderately high dimension, they suggest more research and numerical studies to extend their ideas to truly high-dimensional systems. This paper represents such an extension, with the added element of optimization for robustness.

The SIMPLEX algorithm is an efficient and reliable algorithm for solving Linear Programming (LP) problems.[12,13] Although LP is not popular among dynamics and control engineers, Ref. 14 recognizes LP as a powerful approach to handling a large number of locally linear constraints and proposes solving structure and control optimization problems by introducing a sequence of LP problems. In a successful effort[15] to improve the method in Ref. 14, a different formulation is presented which involves the addition of a local maximum allowable step size constraint and transformations to the standard LP without increasing the dimension of the local LP problem.

We next focus on performance indices that provide some measure of stability robustness of a closed-loop system with respect to inevitable ignorance (of the equations that perfectly model the actual system). As pointed out,[16] the majority of the effort to date has focused on developing analysis tools in the frequency domain, and only a relatively smaller amount of effort has been focused on control law synthesis for robustness. The lack of design-oriented methodology and research is even more pronounced for time domain measures. There is an urgent need to develop design methodology for robust control.

A more direct but perhaps less rigorous class of methods for dealing with the robustness problem is sensitivity minimization. Reference 17 introduces a "modal insensitivity" condition and suggests an eigenstructure assignment approach to achieving modal insensitivity. Recently, Refs. 18 and 19 focused on design algorithms for optimal quadratic regulators with modal insensitivities. However, to impose zero modal sensitivity will prove too restrictive for most applications of high-dimensional systems. In the light of the latter problem, Ref. 15 proposed a direct minimization of the norm of eigenvalue sensitivities.

In summary, the need for a dependable optimization tool that efficiently handles a large number of nonlinear inequality constraints and a high-dimensional design space is critical, particularly for use in simultaneous design of structures and controllers. It is also important that nonlinear optimization iterations be done in such a way that convergence failures are informative, i.e., that suggested restatements of the problem would lead to a least-compromised solution. In addition, methods are needed for incorporating performance and stability robustness tolerances based on recently developed robustness criteria for design of structural control systems.

In Secs. II and III, we present the most specific and immediately useful result, namely, an algorithm involving the sequential LP and continuation method. In Sec. IV, an interpretation of the weighted eigenvalue sensitivity matrix is given. Section V explores the stability robustness criterion of Patel and Toda. In Sec. VI, the concepts of eigenvalue conditioning are applied to derive a familiar criterion that guarantees asymptotic stability. Section VII focuses on applications. Finally, Sec. VIII presents a few concluding remarks.

II. Continuation Approach to Imposing Constraints in Nonlinear Optimization

We consider all sets of constraint vectors that can be reduced to the forms $f(p) \leqq f^o$, $f(p) \geqq f^o$, or $f(p) = f^o$ and indicate all three possibilities by the following notation:

$$f(p)\,\{\leqq,\, =,\, \geqq\}\, f^o \tag{1}$$

where f^o denotes specified objectives and $f(p)$ represents constraint functions whose dependence on parameter vector p is assumed known and well-behaved. The complex nature of constraints generally leads to various problems in the context of mathematical programming. First, the constraint objectives

Received Feb. 13, 1987; revision submitted July 1, 1987.

*Graduate Research Assistant, Engineering Science and Mechanics. Currently, Engineering Specialist, PRC Kentron Inc., Hampton, Virginia. Member AIAA.

†TEES Professor, Department of Aerospace Engineering. Fellow AIAA.

may be such that no feasible solution exists. Second, it may be difficult to locate a starting feasible solution. The continuation method[11] resolves, at least to a significant degree, the preceding problems by seeking out at least a feasible solution to a neighbor of the original problem (if indeed a feasible solution to the originally stated problem does not exist) and providing arbitrarily good initial guesses by starting each iteration with a neighboring converged solution.

The method essentially involves replacing a subset of the original constraint objectives f^o, typically consisting of most compromising and/or demanding constraints, by a sequential neighboring set of constraint objectives $F(\gamma_i)$ where

$$F(\gamma_i) = (1-\gamma_i)f(p^s) + \gamma_i f^o \tag{2}$$

In Eq. (2), p^s is an arbitrarily chosen starting design vector and γ_i a scalar parameter satisfying

$$0 = \gamma_0 \leqq \gamma_1 \leqq \cdots \leqq \gamma_N = 1 \tag{3}$$

For redesign problems, p^s is naturally taken to be the nominal value.[20] The preceding convex combination of starting and final constraint objectives shows that if convergence is achieved at $\gamma = 1$, we recover the original constraint condition in N steps. In most nonlinear problems where linearizations about current values are assumed, of course, the step size of $\Delta\gamma$ can be chosen to validate the assumptions. Perhaps most importantly, failure to reach the final ($\gamma = 1$) solution is softened by convergence to a neighboring solution. The active constraint set and gradient information of the last convergence provides a basis for intelligent revisions of the problem statement. The given approach (with some variations) has been implemented successfully in several problems as documented in Refs. 7, 8, 10, 14, 15, and 20.

III. Optimization via Sequential Linear Programming

The following features motivate the use of sequential linear programming (SLP) for the solution of mathematical programming formulations of simultaneous structure and controller design problems: 1) there is a large number of constraints, mostly inequalities, 2) efficient and reliable LP codes are available, and 3) problem formulations are direct and simple, especially constraint equations. Perhaps further justification for our emphasis on the SLP approach can be attributed to its historical lack of application, even though Refs. 21 and 22 support our conclusion that it is often superior to conventional nonlinear programming methods.

Let us consider the general nonlinear programming problem:

Maximize:

$$J(p) \tag{4}$$

Subject to:

$$f(p)\{\leqq, =, \geqq\} f^o \tag{5}$$

To transform the stated problem to an LP problem, the equations are linearized locally. In some cases, the problem as formulated may be solved iteratively until some type of numerical convergence occurs. However, for general nonlinear problems (where initial guesses close to the optimum are not available and linearization assumptions about a nominal point do not hold over all of the feasible region), additional restrictions are needed for its numerical solution.

For those reasons, we introduce constraints on the maximum parameter corrections allowable locally,

$$-\varepsilon \leqq \Delta p \leqq \varepsilon \tag{6}$$

Equation (6) applies element by element, since Δp and ε are vectors. All elements of ε are assumed positive. Note that the current step size bound, ε, may be adjusted judiciously to accommodate a tradeoff between satisfying the local linearity assumption and permitting a solution to the neighbor constraint that depends on the step change of $(\gamma_i - \gamma_{i-1})$. Some degree of numerical experimentation is usually necessary for an efficient implementation of the given method.

To obtain a nonnegative set of solution vectors without increasing the dimension of the design space, introduce a translational transformation[15]

$$y = \Delta p + \varepsilon \tag{7}$$

In terms of the nonnegative coordinates y, it can be shown[10] that the LP problem at step i can be written as

Maximize:

$$\left[\frac{\partial J}{\partial p}\right]_{p^{i-1}} y \tag{8}$$

Subject to:

$$\left[\frac{\partial f}{\partial p}\right]_{p^{i-1}} y\,\{\leqq, =, \geqq\}\,(1-\gamma_i)f(p^s) + \gamma_i f^o - f(p^{i-1}) + \left[\frac{\partial f}{\partial p}\right]_{p^{i-1}} \varepsilon \qquad y \leqq 2\varepsilon \tag{9}$$

where $p^i = p^{i-1} + y - \varepsilon$, $p^0 = p^s$, $i = 1,\ldots,N$, and y is nonnegative. In Eq. (9), N denotes the number of homotopy steps. By adding slack and/or surplus variables which are themselves nonnegative, the shown LP can be transformed to the standard form[13] and solved using available codes.[25]

IV. Eigenvalue Sensitivity Norm

In this section, the problem of quantifying the sensitivities of eigenvalues due to parameter variations is considered. A direct measure of eigenvalue sensitivity is the quadratic

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\left|\frac{\partial \lambda_i}{\partial p_j}\right|^2 w_{ij} \tag{10}$$

where λ_i and w_{ij} represent eigenvalues and nonnegative weighting factors, respectively. References 10 and 15 demonstrate the feasibility of using such an index to attain low sensitivity while simultaneously assigning eigenvalues. In the sequel, the usefulness of minimizing the direct sensitivity measure of Eq. (10) will be compared to other designs.

To examine the significance of the preceding matrix of eigenvalue sensitivities, the linearly predicted changes in n-eigenvalues due to small changes in m-parameters are written as

$$\Delta\lambda = S\Delta p \tag{11}$$

where S represents n by m matrix of eigenvalue sensitivities. A weighted vector norm of eigenvalue change is defined as

$$\|\Delta\lambda\|^w \triangleq (\Delta\lambda^H W \Delta\lambda)^{1/2}, \qquad W = L^T L \tag{12}$$

where the weight matrix W is a designer-specified positive-definite symmetric matrix that weighs the sensitivity of individual modes. The superscript H represents hermitian transpose,[24] and L is a lower triangular matrix derived from a Cholesky factorization of W. Next, normalize parameter changes Δp by

$$\Delta p = \Theta\delta \tag{13}$$

where Θ is a matrix of normalizing constants. It can be shown[10] that the scalar

$$\sigma_{\max}(L^T S\Theta) = \|L^T S\Theta\|_2 \tag{14}$$

represents an upper bound on the square of weighted eigenvalue error norms, $\|\Delta\lambda\|^w$, for all normalized perturbations satisfying $\delta^T\delta \leqq 1$. Equation (14) represents a convenient scalar index for a weighted measure of eigenvalue sensitivity. For a typical application, L and Θ may be specified by the designer while the elements of S may be iteratively driven so as to minimize the sensitivity measure. The physical significance of weights L and Θ should be noted. It should be emphasized that although both Eqs. (10) and (14) represent sensitivity indices, only Eq. (14) is directly related to a linearly predicted bound on weighted eigenvalue perturbation.

V. Robustness Measure of Patel and Toda

As with any sensitivity formulation, the eigenvalue sensitivity indices discussed in Sec. IV are, strictly speaking, local measures. This implies that for a given configuration with low sensitivity, there is no guarantee that even a small finite perturbation will not destabilize the system. As a result, there is interest in methods that can guarantee various properties of a control system under finite ignorance. In this section, we briefly review an important result due to Ref. 23.

Consider the system described by

$$\dot{x}(t) = [A + E(t)]x(t) \tag{15}$$

where A represents a closed-loop state matrix and the uncertainty of the system is assumed representable by $E(t)$. It can be shown[23] that the system in Eq. (15) remains stable if $E(t)$ satisfies

$$\|E(t)\|_2 \leqq \mu \tag{16}$$

where

$$\mu = 1/\max\,\lambda(P) \tag{17}$$

and P is the solution of the Lyapunov equation

$$A^TP + PA = -2I \tag{18}$$

It can also be shown[23] that

$$\mu \leqq \min[-Re\lambda(A)] \tag{19}$$

Equation (19) gives a bound on achievable robustness with this kind of measure. It is apparent that the ideas of robustness and stability margin are closely related. For pole placement designs by active feedback where eigenvalues are specified, the robustness measure μ can in principle be maximized up to $\min[-Re\lambda(A)]$ by seeking a normal matrix with the desired eigenvalues, provided no additional constraints are imposed. The preceding problem is nontrivial in practice, and there is currently no known algorithm to accomplish this.

VI. Stability Robustness Criterion Using Eigenvalue Bounds

An approach to describing eigenvalue sensitivity is through the concept of conditioning in numerical analysis. In short, the eigenvalue condition number provides an upper bound on the perturbation of the eigenvalues due to a unit norm change in the system matrix.[24]

In this section, the eigenvalue bounding equations are applied to guarantee linear system stability and arrive at a stability robustness criterion. This measure corresponds exactly to the criteria derived in Ref. 23 using the Lyapunov stability theorem and several other lemmas. This is indeed an interesting equivalence. We begin by stating a well-known result in matrix theory:[24] Let A be nondefective, and the eigenvalue and eigenvector matrices be written as

$$\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n), \qquad X = [x_1,\ldots,x_n] \tag{20}$$

If μ and v (where $\|v\| = 1$) is an eigenpair of a perturbed matrix $A + E$, then at least one eigenvalue of A satisfies

$$|\lambda_i - \mu| \leqq \|E\|c(X), \qquad c(X) = \|X\|\|X^{-1}\| \tag{21}$$

where $\|\cdot\|$ represents any fixed 1, 2, or ∞ norms. The significance of the condition number $c(X)$ is clear; it directly establishes the radius of uncertainty within which all eigenvalues are perturbed due to an error E in the A matrix.

To obtain asymptotic stability bounds using the given eigenvalue bound, assume that the unperturbed system A is asymptotically stable. The problem then is to find the upper bound on $\|E\|$ that guarantees that all perturbed eigenvalues remain in the left-half s-plane. Let us consider the set of all E satisfying

$$\|E\|c(X) < \min[-Re\lambda(A)] \tag{22}$$

Then by Eq. (21),

$$\min_i |\lambda_i - \mu_j| < \min[-Re\lambda(A)], \qquad j = 1,\ldots,n \tag{23}$$

Equation (23) states that the distance from every perturbed eigenvalue to the closest unperturbed eigenvalue is always less than the perpendicular distance of the closest unperturbed eigenvalue to the imaginary axis. This implies that all the perturbed eigenvalues will remain in the left-half plane. It follows that for all perturbations satisfying

$$\|E\| < \min[-Re\lambda(A)]/c(X) \tag{24}$$

the system in Eq. (15) remains asymptotically stable. It is clear from Eq. (24) that its right-hand side represents a measure of stability robustness. Note that when A is normal, X is unitary and the condition number takes the minimum value of unity. From Eq. (24) note also that within the set of all A matrices with the same $\min[-Re\lambda(A)]$, the maximum stability robustness condition corresponds to the minimum condition number of the eigenvalue problem, a consistent and intuitively pleasing result! A significance of this is that the problem of minimizing the condition number by eigenvector shaping is equivalent to the problem of maximizing the given stability robustness measure for a fixed set of closed-loop eigenvalues.

A major problem, which is fairly well-known in the literature on robust control, is the conservatism of robustness measures that admit unstructured perturbations. This conservatism is rather democratic in that it afflicts all of the time and frequency domain robustness measures known to the authors. This conservatism is apparently more evident for problems having system matrices that are highly parameterized or that have significant internal structure. In the sequel, the level of conservatism involved is examined numerically for a particular problem.

VII. Applications

A hypothetical structure (see Fig. 1) is chosen for our design study with the equation of motion derived in Ref. 10. The structure consists of a free-free flexible beam with a rigid body attached to the center of the beam by a pin-joint and a torsional spring. This structure can be seen as a planar model of a flexible satellite consisting of a rigid payload and a gimbaled flexible appendage. We consider a 20th-order assumed modes model, and for simplicity we ignore the model reduction issues by assuming our model to be exact. Thus the configuration coordinates are $\{\theta_1,\theta_2,\eta_1,\ldots,\eta_8\}$ where $\eta_i(t)$ is the amplitude of the assumed modes chosen from the first eight modes of the free-free uniform beam.

Four massless torque actuators are assumed, one attached to the mass center of the total structure and the remaining along the flexible beam. For direct output feedback, 12 independent measurements are assumed available: four elastic displacements and velocities at four discrete locations along the beam and angular displacements and velocities of the rigid body and flexible beam frames with respect to inertial frame. Notice that

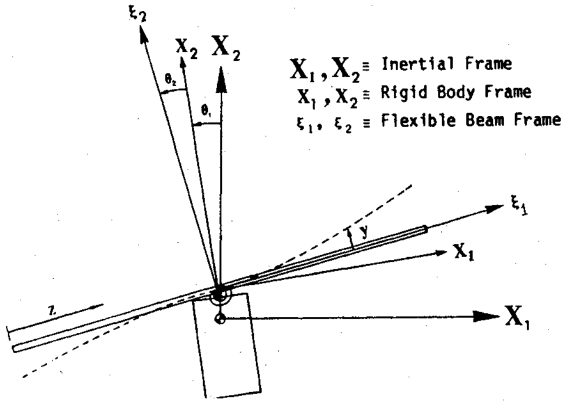

Fig. 1 Rigid body with attached flexible beam.

Table 1 Nominal design variables

Design variable	Symbol	Nominal value
Actuator 1 location	a_1	5 m
Actuator 2 location	a_2	10 m
Actuator 3 location	a_3	15 m
Torsional spring stiffness	k	500 N-m/rad
Thickness of beam	t_F	0.1 m
Young's modulus of beam	E	0.1482×10^9 N/m^2
Mass density of rigid body	ρ_R	300 kg/m^3
Output gain elements $G(i,j)$, $i = 1,...,4; j = 1,...,12$		0

Table 2 Open and desired closed-loop frequencies and damping factors

Mode number	Open-loop[a]		Desired closed-loop[a]	
	ω (rad/s)	ζ	ω^{obj} (rad/s)	ζ^{obj}
1	0.0	0	0.1	0.7
2	0.2803	0.1402×10^{-5}	0.3	0.1
3	0.3443	0.1718×10^{-5}	0.45	0.1
4	1.241	0.6204×10^{-5}	1.0	0.05
5	1.768	0.8839×10^{-5}	1.5	0.05
6	3.981	0.1990×10^{-4}	4.0	0.05
7	5.004	0.2502×10^{-4}	$> \omega^o_{d_6} + 0.1$	0.02
8	8.295	0.4147×10^{-4}	Unconstrained	0.02
9	9.902	0.4951×10^{-4}	Unconstrained	0.02
10	14.34	0.7171×10^{-4}	Unconstrained	0.02

[a] $\lambda_i = \zeta_i\omega_i + j\omega_i(1 - \zeta_i^2)^{1/2}$

the measured local displacements on the beam depend upon all elements in the configuration vector.

Table 1 shows the nominal (or starting) design variables. The open-loop damping factors and frequencies along with desired closed-loop values are given in Table 2. We consider the problem of driving 10 damping factors and 7 natural frequencies to desired values.

A. Optimal Eigenvalue Placement Designs Using Output Feedback

1. Minimum Eigenvalue Sensitivity Design

Since lower-frequency modes generally dominate transient response for structural systems, we choose to minimize the eigenvalue sensitivity of the lower six modes with respect to a set of five design parameters (three actuator locations, torsional stiffness, and beam thickness). The eigenvalue sensitivity cost function is written as

$$\sum_{j=1}^{5} \sum_{i=1}^{6} \left|\frac{\partial \lambda_i}{\partial p_j}\right|^2 w_{ij} \tag{25}$$

where w_{ij} represents the relative weight of the sensitivity of the

Table 3 Lower, upper, and local step size bounds

Parameter	Symbol	
Lower bounds on actuator location	$a_1^\ell, a_2^\ell, a_3^\ell$	0 m
Upper bounds on actuator location	a_1^u, a_2^u, a_3^u	20 m
Lower bounds on spring stiffness	k^ℓ	5 N-m/rad
Lower bound on beam thickness	t_F^ℓ	0.01 m
Upper bound on beam thickness	t_F^u	2 m
Lower bound on beam stiffness	E^ℓ	0.1480×10^9 N/m^2
Lower bound on rigid body density	ρ_R^ℓ	50 kg/m^3
Upper bound on rigid body density	ρ_R^u	1000 kg/m^3
Local step size bounds		
Actuator location	$\Delta a_1, \Delta a_2, \Delta a_3$	0.1 m
Spring stiffness	Δk	30 N-m/rad
Beam thickness	Δt_F	0.01 m
Beam stiffness	ΔE	0.1×10^5 N/m^2
Rigid body density	$\Delta \rho_R$	40 kg/m^3
Gain elements $i = 1,...,4$; $j = 1,...,12$	$\Delta G(i,k)$	10
Frequency separation between modes 7 and 6	$\Delta\omega_{76}$	0.1 rad/s

ith eigenvalue with respect to the jth parameter. In this example, the weights are chosen only to reflect a sum of eigenvalue sensitivity with respect to nondimensionalized parameters, i.e., the weights are taken as the square of the magnitude of the nominal parameters. The set of design parameters used to minimize the given sensitivity measure consists of 7 structural variables and 48 gain elements as shown in Table 1. The constraints are summarized as follows:

Actuator location constraints:

$$a_i^\ell \leqq a_i \leqq a_i^u, \qquad i = 1,2,3 \tag{26}$$

Plant parameter constraints:

$$\begin{gathered} k^\ell \leqq k \\ t_F^\ell \leqq t_F \leqq t_F^u \\ E^\ell \leqq E \\ \rho_R^\ell \leqq \rho_R \leqq \rho_R^u \end{gathered} \tag{27}$$

Eigenvalue constraints:

$$\begin{gathered} \zeta_i = \zeta_i^o, \qquad i = 1,...,10 \\ \omega_{d_i} = \omega_{d_i}^o, \qquad i = 1,...,6 \\ \omega_{d_7} \geqq \omega_{d_6} + \Delta\omega_{76} \end{gathered} \tag{28}$$

Local step size constraints:

$$-\varepsilon_i \leqq \Delta p_i \leqq \varepsilon_i, \qquad i = 1,...,55 \tag{29}$$

where Δp_i and ε_i represent the ith parameter change and the corresponding scalar bounds, respectively. The lower and upper bounds on the actuator locations, structural parameters, and step size are given in Table 3.

The SLP and continuation methods described previously are applied and then solved by an LP code.[25] A starting homotopy step size of $\Delta\gamma = 0.005$ gradually increasing to $\Delta\gamma = 0.1$ was found to be suitable for this problem. After each new increment by $\Delta\gamma$, the value of γ (which corresponds to a percentage enforcement of the eigenvalue relocation constraints) is kept

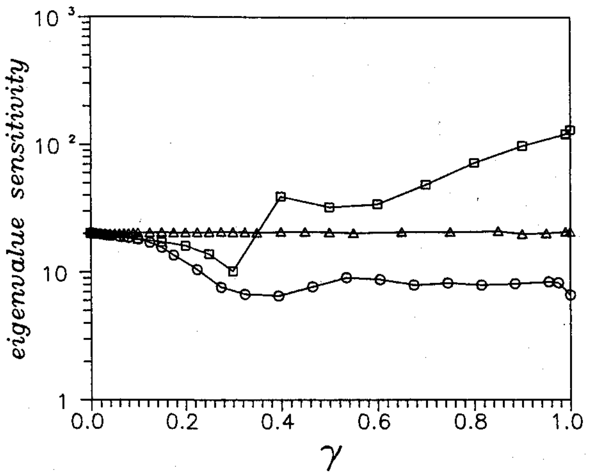

Fig. 2 Convergence histories of eigenvalue sensitivity index.

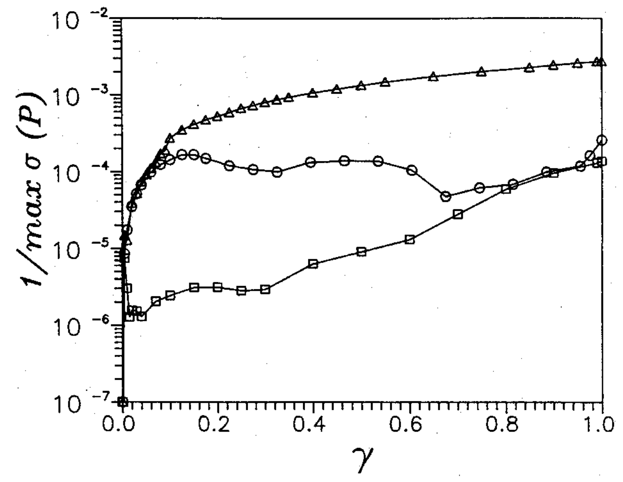

Fig. 3 Convergence histories of stability robustness index.

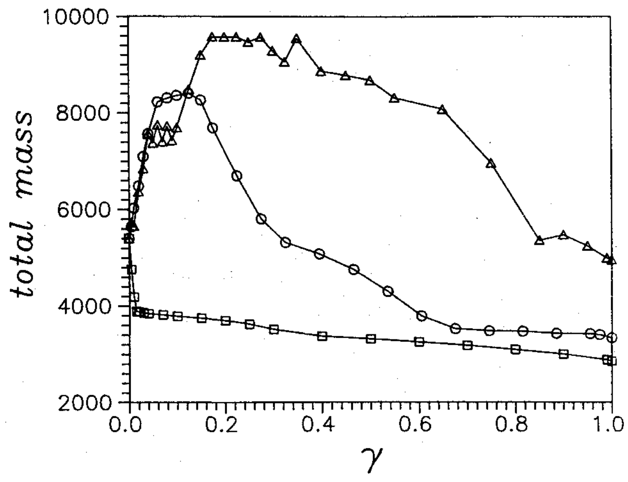

Fig. 4 Convergence histories of total mass.

fixed while the local parameter step size is decreased by one-half and the LP resolved. This is done sequentially until no discernable improvement in cost function is observed.

2. Maximum Stability Robustness Design

Consider the optimization of the stability robustness measure μ given by Eq. (17). Since A is assumed asymptotically stable, P is a symmetric positive-definite matrix, and therefore μ is a well-defined positive number. To formulate the SLP problem, the denominator of the cost function is expanded. The required sensitivity of the maximum eigenvalue of P with respect to the jth parameter is

$$\frac{\partial \bar{\lambda}}{\partial p_j} = \bar{u}^T \frac{\partial P}{\partial p_j} \bar{u} \tag{30}$$

where $\bar{u}$ denotes the real eigenvector corresponding to the maximum eigenvalue $\bar{\lambda}$, which satisfies the real, symmetric eigenvalue problem

$$P\bar{u} = \bar{\lambda}\bar{u} \tag{31}$$

The sensitivity of P as required in Eq. (30) can be obtained by taking the derivative of the Lyapunov equation to get

$$A^T \frac{\partial P}{\partial p_j} + \frac{\partial P}{\partial p_j} A = -\left(\frac{\partial A^T}{\partial p_j} P + P \frac{\partial A}{\partial p_j}\right) \tag{32}$$

From the preceding equation, $\partial P/\partial p_j$ always exists and is unique, since all the eigenvalues of A are assumed to have negative real parts.[26]

3. Minimum Mass Design

The total mass of the structural system is used as the cost function to be minimized. The total mass of the combined structure can be written as

$$M = M_{\text{flexible}} + M_{\text{rigid}} = t_F \, d_F w_F \rho_F + t_R \, d_R w_R \rho_R \tag{33}$$

where the only design variables affecting the total mass are the thickness of the flexible beam t_F and the mass density of the rigid body ρ_R.

B. Numerical Results

Figures 2–4 illustrate the convergence histories of weighted eigenvalue sensitivity, stability robustness, and total mass of the system, respectively. It can be observed that for all three cases, convergence to the desired eigenvalue constraints is complete (i.e., $\gamma = 1.0$ is achieved).

In Fig. 2, eigenvalue sensitivities are plotted. Initially ($\gamma < 0.1$); the minimum mass design results in eigenvalue sensitivity comparable to minimizing the sensitivity directly. For values of $\gamma > 0.1$, the sensitivities fluctuated in an unpredictable manner far above the values of the minimum sensitivity case. Note that maximizing robustness did not influence eigenvalue sensitivity significantly.

Figure 3 shows the stability robustness measures. As expected, the maximum robustness design gives the highest robustness. It is interesting to observe that the sensitivity design results in a robustness history paralleling that of the robustness design history for γ at initial stages ($\gamma < 0.1$) and remains essentially constant at 10^{-4} thereafter. This indicates the usefulness of minimizing eigenvalue sensitivity for robustness optimizations when the eigenvalues are close to the imaginary axis, i.e., when there is a low stability margin. For minimum mass design, the robustness index remains significantly below optimal values. However, the robustness index gradually increases with γ, although its corresponding eigenvalue sensitivity becomes very large. This is probably due to the increase in stability margin with increasing γ. Incidentally, this supports the correct view that stability robustness and sensitivity are not related one-to-one.

The total mass histories in Fig. 4 show large differences between the total mass of each different design. However, the trend of the total mass for sensitivity and robustness designs are similar. Table 4 shows the performance indices at starting and final converged conditions. The improvements in sensitivity, total mass, and robustness are clearly evident. The condition number and an alternate robustness index are shown for additional comparisons.

To further evaluate the robustness and eigenvalue sensitivity of the various designs, an alternate robustness index, Eq. (24),

Table 4 Performance indices at starting and final converged conditions

Cost function	Initial	Final: Sensitivity design	Final: Robustness design	Final: Mass design
Eigenvalue sensitivity	20.0	6.64	20.7	130.6
Total mass	5398	3336	4952	2853
Robustness $(1/\bar{\sigma})$	0.251×10^{-7}	0.258×10^{-3}	0.275×10^{-2}	0.138×10^{-3}
Robustness $\min[-Re\lambda]/c$	0.651×10^{-15}	0.665×10^{-4}	0.314×10^{-3}	0.915×10^{-4}
Condition number c	419.3	453.3	95.9	329.2

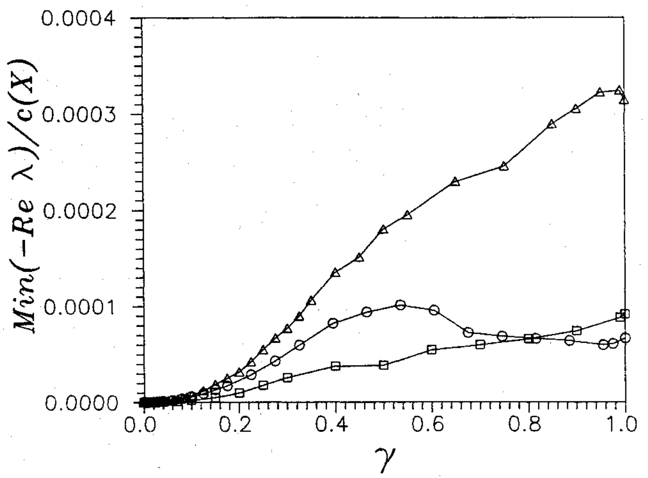

Fig. 5 Convergence histories of stability robustness index.

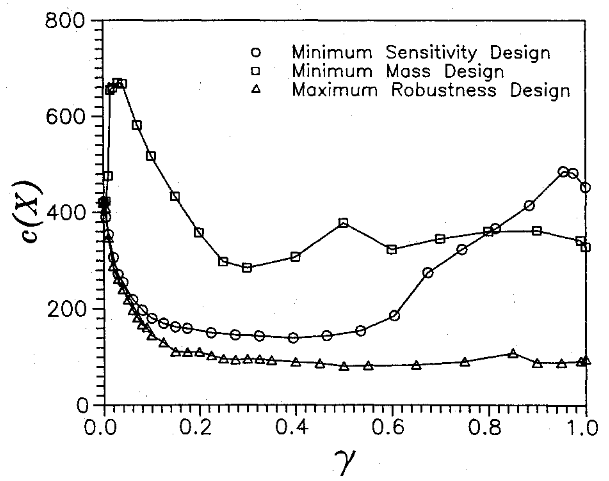

Fig. 6 Convergence histories of eigenvalue condition number.

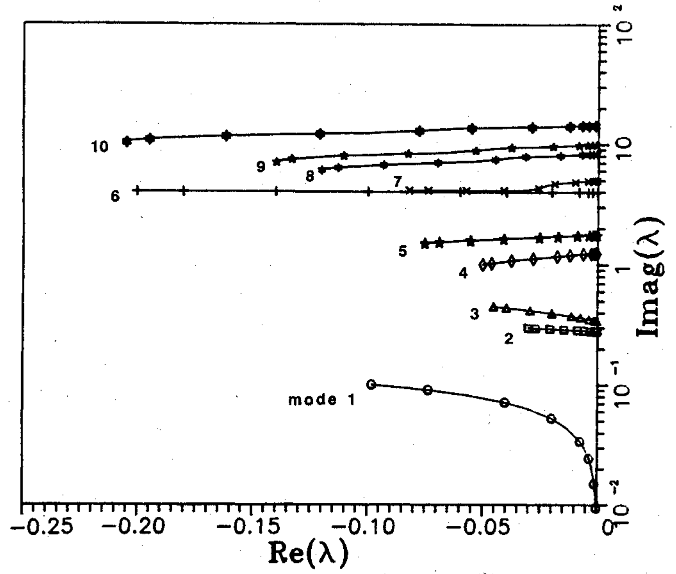

Fig. 7 Closed-loop eigenvalue trajectories with respect to γ.

is plotted along with the eigenvalue condition number in Figs. 5 and 6, respectively. From Fig. 5, the maximum robustness design is clearly seen to be the most robust in terms of the alternative stability robustness index and in fact, Fig. 5 and the corresponding Fig. 3 are very similar. Furthermore, Fig. 6 shows that the condition numbers corresponding to the robustness design case had the smallest values and decreased monotonically to an asymptotic value. Note the interesting similarity between sensitivity and robustness designs with respect to the condition number for small γ (≤ 0.1) from Fig. 6. The stated similarity indicates that the condition number, much like the previously observed eigenvalue sensitivity norm, may serve as a stability robustness index when the system is marginally stable, i.e., for small γ. This observation is not unexpected since the condition number does not contain stability margin information. In other words, the condition number is essentially a local eigenvalue sensitivity measure and as such, it is strongly related to stability robustness only when the eigenvalues lie near the imaginary axis.

Furthermore, observe that the condition number does not appear to be a reliable indicator of stability robustness as compared with the measures of Patel and Toda due to the asymptotic behavior for stability robustness design for $\gamma \gtrsim 0.2$ (Fig. 6), while the other two robustness indices both show a steady increase in robustness over all γ (Figs. 3 and 5).

Figure 7 depicts the closed-loop eigenvalue trajectories of all 10 modes. Much of the parallelism in the eigenvalue trajectories can be attributed to the continuation method of handling constraints; the smooth convergence and the numerical robustness in the trajectories are evident from this figure.

C. Evaluation of True Stability Robustness

It was shown earlier that the robustness measure of Patel and Toda represents a bound on the norm of system matrix perturbation for guaranteeing closed-loop stability. It can be seen that the three designs correspond to three different levels of robustness at $\gamma = 1.0$. For the dual purpose of verifying Patel and Toda's robustness criterion and to obtain an estimate of its conservatism when applied to parameter perturbation problems, we compute true stability limits by sweeping individ-

Table 5 Converged parameters for the three "optimal" designs

Case I: Minimum eigenvalue sensitivity
Case II: Maximum stability robustness design
Case III: Minimum mass design

Case	a_1	a_2	a_3	k	t_F	E	ρ_R
I	4.924	7.878	14.72	1443	0.0841	$0.1480E+9$	51.22
II	4.797	8.924	15.48	1481	0.0939	$0.1496E+9$	262.1
III	4.087	9.795	14.75	2315	0.0709	$0.1494E+9$	50.0

Gain matrices:

$$G_{\mathrm{I}} = \begin{bmatrix} 654.2 & -461.4 & 155.7 & -117.6 & -567.4 & -142.4 & 104.8 & 399.5 & -528.3 & 260.5 & 314.3 & 880.6 \\ -154.2 & -132.9 & 844.6 & -552.6 & -661.6 & -362.1 & -806.8 & 785.8 & 581.0 & -160.4 & 199.4 & -845.5 \\ -614.0 & 54.8 & -316.7 & -179.3 & 524.0 & 283.6 & 320.5 & 124.2 & 261.1 & 354.4 & 232.1 & 191.8 \\ -730.5 & 200.5 & 328.1 & -228.3 & 514.2 & -935.8 & -813.7 & 1060.0 & 539.3 & 323.1 & 193.1 & 221.0 \end{bmatrix}$$

$$G_{\mathrm{II}} = \begin{bmatrix} 137.2 & -944.7 & 523.0 & 202.8 & 720.3 & -573.3 & -1292.0 & -288.3 & -291.2 & 251.1 & 146.2 & 68.9 \\ -530.1 & -874.1 & 811.2 & 442.1 & 822.8 & -1223.0 & -757.5 & 397.4 & -508.8 & -631.3 & 159.6 & -322.2 \\ -478.7 & -710.1 & -1443.0 & -1545.0 & -103.1 & 694.3 & -1202.0 & 410.2 & 1115.0 & 483.6 & 133.5 & -582.4 \\ 363.7 & -497.1 & -32.5 & 7.3 & -1335.0 & -1445.0 & -269.0 & -223.3 & 45.5 & 197.3 & 176.6 & -967.7 \end{bmatrix}$$

$$G_{\mathrm{III}} = \begin{bmatrix} 1038.0 & -411.2 & 980.8 & 93.0 & -319.0 & -535.0 & -934.0 & -374.8 & 455.1 & -1032.0 & 54.5 & -1146.0 \\ -71.2 & -157.3 & 753.3 & -1051.0 & -703.4 & -282.2 & 823.4 & -928.3 & 285.2 & 37.9 & -549.3 & 321.6 \\ -611.7 & -198.3 & 302.2 & 210.1 & 10.9 & 1218.0 & -1106.0 & 1165.0 & 13.9 & 251.5 & 516.7 & -654.6 \\ -707.5 & -145.7 & 469.1 & -57.3 & 1194.0 & -1166.0 & -404.0 & -683.8 & -170.9 & 220.0 & 558.2 & 353.1 \end{bmatrix}$$

Table 6 True stability limits for individual parameters

	Percentage perturbation for instability		
Parameter	Sensitivity design	Robustness design	Mass design
Actuator 1 location[a]	5.90	7.68	2.87
Actuator 2 location[a]	7.09	6.25	1.56
Actuator 3 location[a]	5.80	6.20	3.60
Torsional stiffness[b]	16.6	296	23.2
Beam thickness[b]	4.0	9.4	2.1
Young's modulus[b]	14.0	31.3	8.0
Mass density[b]	4.1	61.2	1.0
Gain element (1, 1)[c]	11.7	54.9	20.6
Gain element (1, 3)[c]	31.1	36.6	29.4
Gain element (2, 8)[c]	70.6	715	55.7
Gain element (3, 9)[c]	52.2	66.8	18.9
Gain element (4, 11)[c]	27.0	124	26.8
Average perturbation	20.8	118	16.1
Average $\|E\|_2$	3.32	9.83	1.92

[a]Normalized by 10. [b]Normalized by nominal values (Table 1).
[c]Normalized by 500.

ual parameters while rigorously (nonlinearly) calculating the corresponding eigenvalues' loci to detect the actual onset of instability. Table 5 shows the converged design variables at $\gamma = 1.0$ where the parametric sweeping begins. In Table 6, the stability limits of a selected set of 12 individual parameters (determined by the first eigenvalue crossing the imaginary axis) are shown. It can be concluded from this table that the maximum robustness design tolerates the largest amount of parameter perturbations, i.e., that the maximally robust design is the most robust of the three designs. On the other hand, the minimum mass design generally tolerates the least amount of parameter perturbation, i.e., it is the least robust of the three designs.

The conservatism of the robustness measure of Eq. (17) in guaranteeing closed-loop stability can also be seen from Table 6. All three averages of the 12 individually perturbed matrix norms are greater by several orders of magnitude than the predicted bounds that guarantee stability (see Table 4). It should be noted that the preceding results are apparently typical; the predicted matrix norm bounds that guarantee stability are quite often a few orders of magnitude smaller than true stability limits. This large degree of conservatism is not unexpected for this type of robustness measure since a single scalar measure of perturbation magnitude in a multidimensional parameter space (in our case, of dimension 55) is bound to be highly conservative, not to mention the heavily structured nature of the parameterized perturbations introduced here, whereas unstructured perturbations are assumed in the Patel/Toda theory. Nevertheless, we conclude from the stated results that a significant numerical difference in the given robustness measure corresponds to a significant difference in the actual stability robustness of the closed-loop system.

The preceding numerical results confirm the usefulness of the robustness measure as an objective function for robustness optimization but does not resolve the problem that this measure (as well as all known robustness measures) is overly conservative if used as a predictive measure of the size of the perturbation that will lead to actual instability. The fortunate paradox is that maximization of this conservative measure is demonstrated to be very effective in substantially increasing the system's true robustness.

VIII. Concluding Remarks

We have presented and demonstrated a novel design algorithm for numerical applications using stability robustness measures, which have not received much historical attention. Three different cost functions (total mass, stability robustness, and eigenvalue sensitivity) have been successfully optimized with respect to a set of design parameters that included structural and control parameters and actuator locations. It was found that some similarity in the convergence histories during gradual imposition of the constraints exists between eigenvalue sensitivity and stability robustness designs. The attractive practical consequences of optimizing the robustness criterion were established in spite of its conservatism when used as a predictive bound on allowable perturbation in the system matrix. It can be concluded that a significant improvement in the robustness measure does indeed correspond to a significant improvement in the actual stability robustness of the closed-loop system.

Acknowledgments

This work was supported in part by Air Force Office of Scientific Research Contract F49620-86-K-0014DEF. The technical liaison of Dr. A. K. Amos of AFOSR is appreciated.

References

[1]Balas, M. J., "Trends in LSS Control Theory: Fondest Hopes, Wildest Dreams," *IEEE Transactions on AC*, Vol. AC-27, No. 3, June 1982, pp. 522–535.

[2]Bekey, I. and Naugle, J. E., "Just Over the Horizon in Space," *Astronautics and Aeronautics*, May 1980, pp. 64–76.

[3]Khot, N. S. et al., "Optimal Structural Modifications to Enhance the Optimal Active Vibration Control of Large Flexible Structures," *26th Structures, Structural Dynamics, and Materials Conference*, Orlando, FL, April 15–17, 1985.

[4]Hale, A. L., Lisowski, R. J., and Dahl, W. E., "Optimal Simultaneous Structural and Control Design of Maneuvering Flexible Spacecraft," *Journal of Guidance and Control*, Vol. 8, Jan.–Feb. 1985, pp. 86–93.

[5]Salama, M., Hamidi, M., and Demsetz, L., "Optimization of Controlled Structures," *JPL Workshop on Identification and Control of Flexible Space Structures*, San Diego, CA, 1984.

[6]Venkayya, V. B. and Tischler, V. A., "Frequency Control and the Effect on the Dynamic Response of Flexible Structures," *AIAA Journal*, Vol. 22, Sept. 1984, pp. 1293–1298.

[7]Bodden, D. S. and Junkins, J. L., "Eigenvalue Optimization Algorithms for Structural/Control Design Iterations," *American Controls Conference*, San Diego, CA, June 6–8, 1984.

[8]Junkins, J. L., Bodden, D. S., and Turner, J. D., "A Unified Approach to Structure and Control System Design Iterations," *Fourth International Conference on Applied Numerical Modelling*, Tainan, Taiwan, Dec. 27–29, 1984.

[9]Haftka, R. T. et al., "Sensitivity of Optimized Control Systems to Minor Structural Modifications," *26th Structures, Structural Dynamics, and Materials Conference*, Orlando, FL, April 15–17, 1985.

[10]Lim, K. B., "A Unified Approach to Structure and Controller Design Optimizations," Ph.D Dissertation, Virginia Polytechnic Institute and State Univ., Blacksburg, VA, 1986.

[11]Dunyak, J. P., Junkins, J. L., and Watson, L. T., "Robust Nonlinear Least Squares Estimation Using the Chow-Yorke Homotopy Method," *Journal of Guidance and Control*, Vol. 7, No. 6, 1984, pp. 752–754.

[12]Dantzig, G. B., *Linear Programming and Extensions*, Princeton Univ. Press, 1963.

[13]Hadley, G., *Linear Programming*, Addison-Wesley, Reading, MA, 1962.

[14]Horta, L. G., Juang, J.-N., and Junkins, J. L., "A Sequential Linear Optimization Approach for Controller Design," AIAA Paper 85-1971-CP, Aug. 1985.

[15]Lim, K. B. and Junkins, J. L., "Minimum Sensitivity Eigenvalue Placement via Sequential Linear Programming," *Proceedings of the Mountain Lake Dynamics and Control Institute*, edited by J. L. Junkins, Mountain Lake, VA, June 9–11, 1985.

[16]Newsom, J. R. and Mukhopadhyay, "A Multiloop Robust Controller Design Study Using Singular Value Gradients," *Journal of Guidance and Control*, Vol. 8, No. 4, July–Aug. 1985, pp. 514–519.

[17]Howze, J. W. and Cavin, R. K., "Regulator Design with Modal Insensitivity," *IEEE Transactions on AC*, Vol. AC-24, No. 3, June 1979, pp. 466–469.

[18]Raman, K. V., "Modal Insensitivity with Optimality," Ph.D Dissertation, Drexel Univ., Philadelphia, PA, 1984.

[19]Raman, K. V. and Calise, A. J., "Design of an Optimal Output Feedback Control System with Modal Insensitivity," AIAA Paper 84-1940, Aug. 1984.

[20]Lim, K. B. and Junkins, J. L., "Optimal Redesign of Dynamic Structures via Sequential Linear Programming," *Fourth International Modal Analysis Conference*, Los Angeles, CA, Feb. 3–6, 1986.

[21]Palacios-Gomez, F., Lasdon, L., and Engquist, M., "Nonlinear Optimization by Successive Linear Programming," *Management Science*, Vol. 28, No. 8, Oct. 1982, pp. 1106–1120.

[22]Palacios-Gomez, F., "The Solution of Nonlinear Optimization Problems Using Successive Linear Programming," Ph.D Dissertation, Univ. of Texas, Austin, TX, 1980.

[23]Patel, R. V. and Toda, M., "Quantitative Measures of Robustness for Multivariable Systems," TP8-A, *Proceedings of JACC*, San Francisco, CA, 1980.

[24]Noble, B. and Daniel, J. W., *Applied Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

[25]IMSL Reference Manual, International Mathematical and Statistical Library, 1982.

[26]Chen, C. T., *Introduction to Linear Systems Theory*, Holt, Rinehart & Winston, 1970.